Membrane paradigm
Master internship

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1 Introduction

Black holes are among the most amazing phenomena of nature. Though they were first seen as solutions to Einstein's field equations, physicists have recently been able to take a picture of them,see *Figure 1* [1]. Comparing the properties of black holes with the laws of thermodynamics opens up more intriguing aspects of black holes. Before delving further, let us briefly discuss some of the weirdest properties of black holes.

Light cones tipping over : When we talk about light cones tipping over near a black hole's event horizon, we're describing how the extreme curvature of spacetime alters the paths that light and other objects can take. As an observer gets closer to the black hole, the light cones representing the possible paths of light and causality start to tilt towards the black hole's center. This tilt indicates that once you cross the event horizon, all paths inevitably lead towards the black hole's singularity, making escape impossible. In short, the tipping over of light cones illustrates how the strong gravitational pull of a black hole warps the fabric of spacetime, fundamentally changing the rules of how light and objects move near it.

Infinite red shift : It comes from the extreme gravitational effects near event horizon of a black hole. When an object approaches horizon, the gravitational attraction becomes incredibly strong. This gravitational attraction causes light emitted from the object to lose energy as it tries to escape the intense gravitational field. As a result, the wavelength of the light increases, shifting it towards the red end of the electromagnetic spectrum. This phenomenon is known as gravitational redshift (exactly like a doppler effect but in a gravitationnal case). As the object gets closer to the event horizon, the gravitational redshift becomes more and more important. Near the horizon, the gravitational attraction is so intense that the redshift becomes infinite. This means that the light emitted from the object would appear infinitely redshifted to an observer located far away from the black hole. In practical terms, this infinite redshift near the event horizon implies that any information or light emitted from an object nearing the event horizon would become undetectable to external observers. It effectively represents a point of no return, beyond which no information or light can escape from the gravitational grip of the black hole. This precede the light cones tipping over and is obviously related to it.

Singular jacobians : The Jacobian determinant is a mathematical concept used in calculus to describe how coordinate transformations affect volume elements. Near

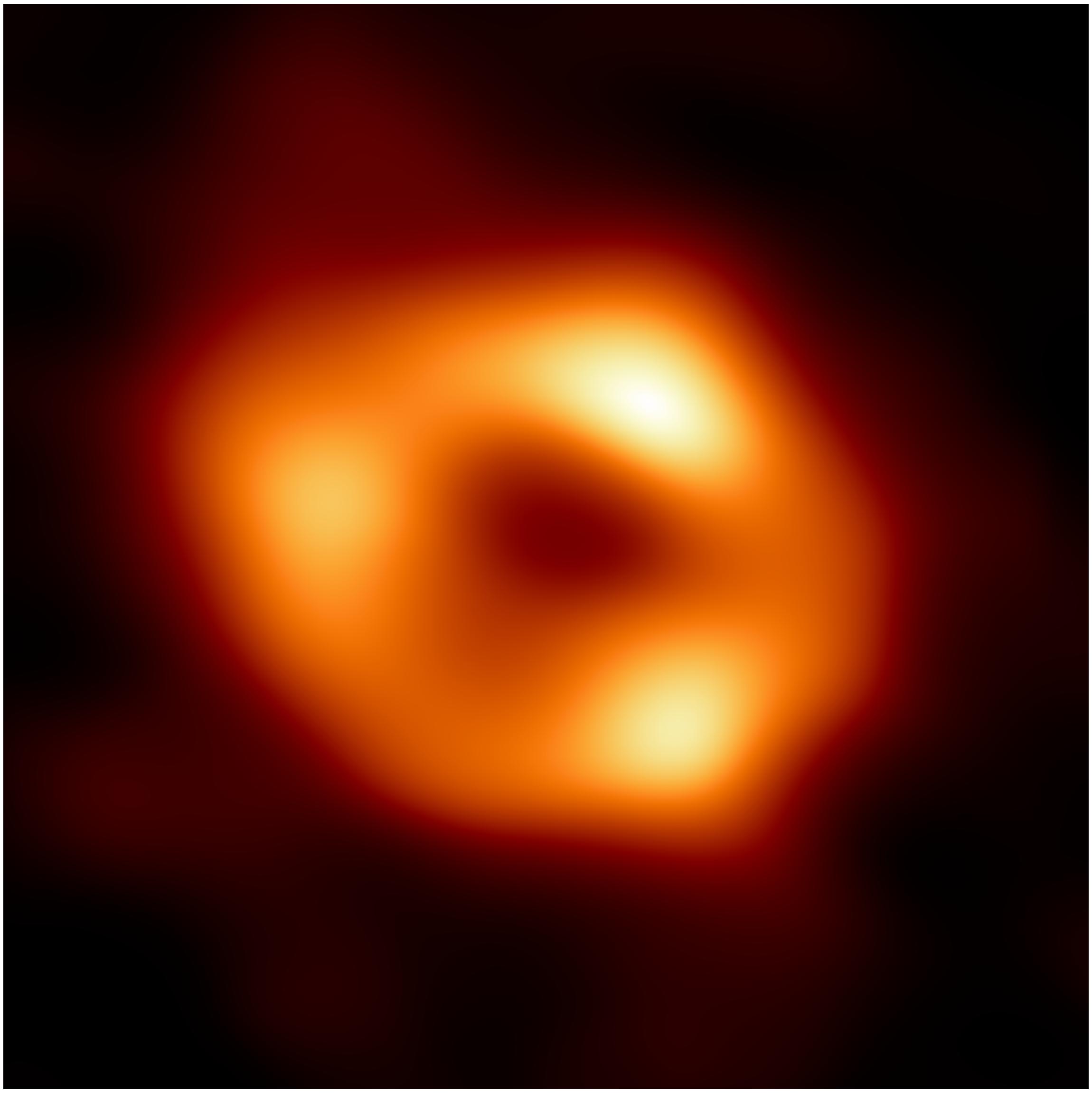

Figure 1: First image of the black hole at the center of the Milky Way.

the horizon, the region is higly curved and the coordinate systems may become highly distorted, leading to complications in calculations involving changes of coordinates. When the Jacobian determinant becomes singular, it means that the transformation between coordinate systems is not well-behaved or breaks down entirely. The mathematics describing the geometry of spacetime encounters difficulties, often indicating the breakdown of classical theories like general relativity. These singularities in the Jacobian can be indicative of the need for more advanced theories, such as quantum gravity, to fully understand the physics at play in these extreme environments. We will obviously not try to make quantum gravity here.

A vector that is both normal and tangent to event horizon : This property takes place only for null hypersurface (our event horizon). In this case the normal verctor has a null norm. But the norm of a vector is also the scalar product of the vector with itself, meaning that, because the scalar product (the norm) is zero, the normal vector is orthogonal to itself. So, the normal vector is also tangent to the null hypersurface (the event horizon).

With all these peculiarities, it is challenging to theoretically model a black hole in all its glory. Therefore, physicists employ different tricks/methods to develop alternative theories that preserve all of the physics but are able to bypass at least some of these peculiarities The main goal of this internship is to compute some properties of Black Holes (BH) with an effective model, called the '**Membrane Paradigm**', introduced in [4].

This model is relatively simple. It permits the calculation of the exterior properties of a black hole without using quantum mechanics. The method is based on the peculiarity of the event horizon. In the literature, it's evident that the event horizon can be understood as following equations akin to those describing a fluid bubble. This fluid bubble exhibits shear and bulk viscosities as well as electrical conductivity. This is exactly what we will exploit to compute the properties of the black hole. Additionally, it's possible to define local surface densities, like charge or energy-momentum, on the bubble surface, which follow conservation laws. Remarkably, within general relativity, equations for the horizon closely resemble Ohm's law, the Joule heating law, and the Navier-Stokes equation, even for arbitrary non-equilibrium black holes.

Now, you might be asking yourself, 'great, but how does it work?' We consider what we call a 'stretched horizon,' which is a surface just outside the true event horizon. Why? Because there is no singularity here. So, we can fix a spatial coordinate (the radial one in spherical coordinates) and we make a 2+1 split of the three other coordinates (The global split is called the '2+1+1 split of the stretched horizon'). Due to its non-singular induced metric, the stretched horizon offers a more manageable boundary for anchoring external fields. Beyond a complex boundary layer, the equations governing the stretched horizon closely approximate those for the actual horizon. This conceptualization of a black hole as a dynamic, time-like surface, akin to a membrane, is referred to as the membrane paradigm.

Because nothing can emerge from a black hole, the equations of motion (EOM) have to follow from the variation of an action outside the black hole. We need to be careful about the cancellation of the EOM on the stretched horizon. In general, it does not cancel. Mathematically, we must add a surface term that cancels this term 'by hand' to obtain the correct EOM. In order to derive the complete equations governing motion through extremizing an action, it's insufficient merely to set the bulk variation of the action to zero. Additionally, it's imperative to incorporate the boundary conditions. In this context, we impose Dirichlet boundary conditions, specifically $\delta\phi = 0$, at both the singularity and the spacetime boundary, where ϕ represents any field.

$$S_{world} = (S_{out} + S_{surf}) + (S_{in} - S_{surf}) \tag{1.1}$$

We clearly see that $\delta S_{in} - \delta S_{surf} = 0$ when we vary the action and when we set $\delta S_{out} + \delta S_{surf} \equiv 0$. So we have broken the action into two parts, both stationnary on their own and we do not need any new boundary conditions. Physically, we interpret this added surface term as gravtitionnal and electromagnetic (fictional) sources on the stretched horizon. To enhance clarity in the report, we analyze the electromagnetic and gravitational boundary terms separately.

To ensure consistency, we will apply our results to an Reissner-Nordström (RN) black hole as an example, and see if the results of our method are the same as the already known results for an RN black hole (temperature, entropy,...). We will finish by giving future directions of the method and explain how can it be useful for the black hole Physics.

2 Conventions and notations

In this section, we will introduce all the conventions and the notations we will use for the computations. These are essentially the same as the conventions in the reference paper [4]. We use a positive signature (-,+,+,+) for spacetime metric and geometrized units, which means that $G \equiv c \equiv 1$. In [5], it is shown that there exists a unique null generator, which we denote as l^μ, at every point of the true (event) horizon of the black hole. The parameterization of this generator can be done using a regular time coordinate. We normalize it by fixing the time-at-infinity and making it equal to the norm. We previously discussed what we referred to as the 'stretched horizon', denoted as $\mathcal{H}$. It is positioned just outside the true horizon of the black hole. By convention, it is considered a time-like surface, and its location is parameterized by $\alpha \ll 1$ to ensure that the true horizon and the stretched horizon coincide in the limit as $\alpha \to 0$. As mentioned in the introduction, the stretched horizon allows us to formulate a conventional action because the metric on a time-like surface is non-degenerate, whereas it is degenerate for a null surface, including the true horizon

As mentionned in [4], we regard the stretched horizon as the world-tube of time-like observers just outside the true horizon. These observers are called 'fiducial' and, although extreme in their properties, provide practical measurements. The stretched horizon, acting as a surrogate, is more accessible for external observation compared to the true horizon.

In this report, $\mu, \nu, ... = 0, 1, 2, 3$ for spacetime co-ordinates. And we use $A, B, ... = 2, 3$ for space co-ordinate on $\mathcal{H}$. We note n^μ the normal vector congruence on $\mathcal{H}$. It is spacelike, unit and outward-pointing. We also take, for fiducial observers, the world lines U^μ which we parametrize by their proper time τ. So α is defined to have $\alpha U^\mu \to l^\mu$ and $\alpha n^\mu \to l^\mu$ so that the null generator is both tangential and normal to the event horizon. This is a property of null surfaces [5]. Hence $g_{\mu\nu}$ is the spacetime 4-metric, $h_{\mu\nu}$ is the 3-metric on $\mathcal{H}$ viewed as a 4-dimensional tensor written in terms of the spacetime 4-metric and the normal vector. It means the h^μ_ν projects to the 3-tangent space from the spactime tangent space. We can make a 2+1+1 split of spacetime by defining, in the same way, the 2-metric, γ_{AB}, of the spacelike section of $\mathcal{H}$ to which U^μ is normal, in terms of the stretched horizon 3-metric and U^μ.

We introduce three covariant derivative here. The classical 4-covariant derivative ∇_μ, the 3-covariant derivative $_{|\mu}$ and the 2-covariant derivative $_{||\mu}$. For a vector within the stretched horizon, the covariant derivatives are linked by the expression $h^\rho{}_\mu \nabla_\rho w^\nu = w^\nu_{|\mu} - K^\rho{}_\nu w_\rho n^\nu$, where $K^\nu_\mu \equiv h^\rho_\mu \nabla_\rho n^\nu$ denotes the extrinsic curvature of the stretched horizon, known as the second fundamental form [5]. We can make a summary

$$U^\mu = \left(\frac{d}{d\tau}\right), U^2 = -1, \lim_{\alpha \to \infty} \alpha U^\mu = l^\mu \tag{2.1}$$

$$l^2 = 0 \tag{2.2}$$

$$n^2 = +1, a^\mu = n^\nu \nabla_\nu n^\mu, \lim_{\alpha \to \infty} \alpha n^\mu = l^\mu \tag{2.3}$$

$$h^\mu_\nu = g^\mu_\nu - n^\mu n_\nu, \gamma^\mu_\nu = h^\mu_\nu + U^\mu U_\nu = g^\mu_\nu - n^\mu n_\nu + U^\mu U_\nu \tag{2.4}$$

$$K^\nu_\mu \equiv h^\rho_\mu \nabla_\rho n^\nu, K_{\mu\nu} = K_{\nu\mu}, K_{\mu\nu} n^\nu = 0 \tag{2.5}$$

3 Membrane at the Horizon

3.1 Electromagnetic

3.1.1 Equations of Motion

$$S_{out}[A_\nu] = \int d^4x\sqrt{-g}\left(\frac{-1}{16\pi}F^2 + J\cdot A\right) \tag{3.1}$$

Where F is the electromagnetic field strength and A^μ is the potential vector. To do the variation of this action, we need to recall some expressions:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{3.2}$$

We have a covariant derivative because we are in general relativity. Let's do the variation of the action (3.1), with respect to A_μ here.

$$\begin{aligned}\delta S_{out}[A_\nu] &= \int d^4x\sqrt{-g}\ \delta\left(\frac{-1}{16\pi}F^2 + J\cdot A\right)\\ &= \int d^4x\sqrt{-g}\left(\frac{-1}{16\pi}\delta(F^2) + \delta(J\cdot A)\right)\end{aligned} \tag{3.3}$$

Let's focus on the two terms separitely. We start with:

$$\begin{aligned}\delta(F^2) &= \delta(F^{\mu\nu}F_{\mu\nu})\\ &= F^{\mu\nu}\ \delta\left(F_{\mu\nu}\right) + \delta\left(F^{\mu\nu}\right)F_{\mu\nu}\\ &= F^{\mu\nu}\ \delta\left(F_{\mu\nu}\right) + F_{\mu\nu}\ \delta\left(g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right)\\ &= F^{\mu\nu}\ \delta\left(F_{\mu\nu}\right) + F^{\rho\sigma}\ \delta\left(F_{\rho\sigma}\right)\\ &= 2F^{\mu\nu}\ \delta\left(F_{\mu\nu}\right)\\ &= 2\left(\nabla_\mu\left(\delta A_\nu\right) - \nabla_\nu\left(\delta A_\mu\right)\right)F^{\mu\nu}\\ &= 2\left(\nabla_\mu\left(\delta A_\nu\right)F^{\mu\nu} - \nabla_\nu\left(\delta A_\mu\right)F^{\mu\nu}\right)\\ &= 4\ \nabla_\mu\left(\delta A_\nu\right)F^{\mu\nu}.\end{aligned} \tag{3.4}$$

To derive the sixth line from definition (3.2), and the eighth line from renaming $\mu \leftrightarrow \nu$ in the second term and using the antisymmetry of the Faraday tensor. Now, let's focus on the second term of (3.3):

$$\delta\left(J\cdot A\right) = J^\nu\ \delta A_\nu \tag{3.5}$$

We can now put (3.4) and (3.5) in (3.3) to have the complete expression of the variation of action:

$$\delta S_{out}[A_\nu] = \int d^4x\sqrt{-g}\left(\frac{-1}{4\pi}\nabla_\mu(\delta A_\nu)F^{\mu\nu} + J^\nu\ \delta A_\nu\right). \tag{3.6}$$

By using the leibniz rule on the first term in the parentheses, we can rewrite equation (3.6):

$$\delta S_{out}[A_\nu] = \int d^4x\sqrt{-g}\left[\left(\frac{1}{4\pi}\nabla_\mu F^{\mu\nu} + J^\nu\right)\delta A_\nu\right] - \frac{1}{4\pi}\int d^4x\sqrt{-g}\nabla_\mu\left(\delta A_\nu F^{\mu\nu}\right). \tag{3.7}$$

The first term leads to the equations of motion (EOM) by setting $\delta S \approx 0$ and the second term is a boundary term.
Since δA_ν is arbitrary, setting $\delta S \approx 0$ results in :

$$\nabla_\mu F^{\mu\nu} = 4\ \pi J^\nu. \tag{3.8}$$

Where we have used the antisymmetry of the Faraday tensor. That is the well-known equations of motion of electromagnetic case.

3.1.2 Surface action

Now we can focus on the boundary term. We recall the Stokes's theorem [2]:

$$\int_M d^nx\ \sqrt{-g}\ \nabla_\mu V^\mu = \int_{\partial M} d^{n-1}y\ \sqrt{-h}\ n_\mu V^\mu. \tag{3.9}$$

Where n_μ is the unit normal to the boundary. In our case, if we take the boundary term, we have:

$$\frac{1}{4\pi}\int d^4x\sqrt{-g}\ \nabla_\mu\left(\delta A_\nu F^{\mu\nu}\right) = \frac{1}{4\pi}\int d^3x\sqrt{-h}\ n_\mu F^{\mu\nu}\delta A_\nu. \tag{3.10}$$

That means that the surface term is just :

$$\frac{1}{4\pi}\int d^3x\sqrt{-h}\ F^{\mu\nu}n_\mu\delta A_\nu. \tag{3.11}$$

Here h is the determinant of the induced metric, n_μ is the outward-pointing space-like unit normal to the stretched horizon. As said before, we want this term to cancel. We now must add the following term to the action :

$$S_{surf}[A_\nu] = \int d^3x\sqrt{-h}\ j_s \cdot A. \tag{3.12}$$

Where $j_s^\mu = \frac{1}{4\pi} F^{\mu\nu} n_\nu$ is the surface 4-current. Its time component is σ, a surface charge and its spatial components, $\vec{j_s}$, is a surface current.
Recalling the expression of the Faraday tensor (in cartesion co-ordinate):

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix},$$

we can take the time component of j_s^μ :

$$j_s^0 = \frac{1}{4\pi} F^{0\nu} n_\nu = \frac{-1}{4\pi} \vec{E} \cdot \vec{n} = \frac{-1}{4\pi} E_\perp. \tag{3.13}$$

But we have said that the time component of the surface 4-current j_s^μ is σ. And because, in our case, $U_\mu = (1, 0, 0, 0)$, we can write $F^{0\nu} = F^{\mu\nu} U_\mu$ and just write, with (3.13):

$$E_\perp = -U_\mu F^{\mu\nu} n_\nu = 4\pi\sigma. \tag{3.14}$$

We can perform similar computations for the spatial components, we compute the components of $\vec{j_s}$ by using the definitions of the surface 4-current and of the Faraday tensor. We arrive at:

$$\vec{j_s}^A = \frac{1}{4\pi} \left(\vec{B} \times \vec{n} \right)^A. \tag{3.15}$$

Wich leads, by taking the cross product wiht $\vec{n}$ to:

$$\vec{B}_{||}^A = 4\ \pi \left(\vec{j_s} \times \vec{n} \right)^A \tag{3.16}$$

The membrane paradigm is distinguished by the fact that σ and $\vec{j_s}$ represent local densities, implying that the aggregate charge of the black hole can be determined by integrating σ across the membrane's surface at a fixed universal time. Now, let's take the divergence of the surface 4-current j_s^μ:

$$\begin{aligned} \nabla_\mu j_s^\mu &= \nabla_\mu \left(\frac{1}{4\pi} F^{\mu\nu} n_\nu \right) \\ &= -J^n. \end{aligned} \tag{3.17}$$

We have arrived at the last line by using the EOM (3.8) and by defining $J^n = J^\mu n_\mu$. But on we also have :

$$\begin{aligned} \nabla_\mu j_s^\mu &= \nabla_0 j_s^0 + \nabla_i j_s^i \\ &= \partial_0 \sigma + j_{||A}^A \\ &= \partial_0 \sigma + (j_s^\mu \gamma_\mu^A)_{||A}. \end{aligned} \tag{3.18}$$

The first line is obtained just by splitting the time and the spatial components. The first term is obtained just because σ is a scalar while the second term is obtained by using the fact that by (3.16) j_s^i is related to $\vec{B}_{||}$ wich means that it is parallel to the stretched horizon and so we can only the component $j^A_{||A}$ is not null. The last line is just a rewrite of the previous line.
Now, we define the "two dimensional divergence of the membrane surface current" :

$$\vec{\nabla}_2 \cdot \vec{j_s} \equiv \left(\gamma_a^A j_s^a\right)_{||A} . \tag{3.19}$$

And by inserting it into (3.18), and equal with (3.17), we have:

$$-J^n = \vec{\nabla}_2 \cdot \vec{j_s} + \frac{\partial \sigma}{\partial \tau} \tag{3.20}$$

since the time component is parametrized by the proper time τ.
We can now give a physical meaning for J^n : this is the charge density entering the hole per unit area per unit proper time, τ. This equation essentially embodies the principle of local charge conservation, suggesting that any charge entering the black hole effectively remains on the membrane. In essence, the membrane acts as an impermeable barrier to charge influx.

As mentionned in [4], the equations we have thus far are sufficient for determining the fields beyond the horizon, based on initial conditions outside it. Plausible initial conditions at the horizon would involve finite fields as observed by freely-falling observers (that we not FFO's) near the stretched horizon. Unlike inertial observers crossing the horizon without encountering any curvature singularity, fiducial observers (that we note FIDO's) conducting measurements at the membrane experience infinite acceleration. This renders their measurements singular due to the effects of infinite Lorentz boosts. Let's begin by writing the Lorentzian transformation taken (in orthonormal co-ordinate):

$$\begin{pmatrix} E_\theta^{FIDO} \\ B_\phi^{FIDO} \\ E_\phi^{FIDO} \\ B_\theta^{FIDO} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & \gamma & -\gamma\beta \\ 0 & 0 & -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_\theta^{FFO} \\ B_\phi^{FFO} \\ E_\phi^{FFO} \\ B_\theta^{FFO} \end{pmatrix} . \tag{3.21}$$

Where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Computing component by component:

$$\begin{cases} E_\theta^{FIDO} = \gamma(E_\theta^{FFO} - \beta B_\phi^{FFO}) \\ B_\phi^{FIDO} = \gamma(B_\phi^{FFO} - \beta E_\theta^{FFO}) \\ E_\phi^{FIDO} = \gamma(E_\phi^{FFO} - \beta B_\theta^{FFO}) \\ B_\theta^{FIDO} = \gamma(B_\theta^{FFO} - \beta E_\phi^{FFO}) \end{cases} \tag{3.22}$$

And because the fiducial observers are infinitely accelerated, $\frac{v}{c} = v \approx 1$ (Here $v = 1$ because we use geometrized units), $\gamma >> 1$. So we can rewrite (3.22):

$$\begin{cases} E_\theta^{FIDO} \approx \gamma(E_\theta^{FFO} - B_\phi^{FFO}) \\ B_\phi^{FIDO} \approx \gamma(B_\phi^{FFO} - E_\theta^{FFO}) \\ E_\phi^{FIDO} \approx \gamma(E_\phi^{FFO} - B_\theta^{FFO}) \\ B_\theta^{FIDO} \approx \gamma(B_\theta^{FFO} - E_\phi^{FFO}) \end{cases} \tag{3.23}$$

Here, the orthonormal coordinates are those parallel to the stretched horizon, as we have fixed the radial and time coordinates. Therefore, we can show that it can be written:

$$\vec{E}_\parallel^{FIDO} = \hat{n} \times \vec{B}_\parallel^{FIDO}. \tag{3.24}$$

This equation describes the 'regularity condition,' which simply states that a black hole behaves like a perfect absorber, meaning that all radiation in the normal direction is ingoing.

We see that if we combine (3.16) with (3.24), we obtain the equation:

$$\begin{aligned} \vec{E}_\parallel &= \hat{n} \times \vec{B}_\parallel \\ &= \hat{n} \times 4\pi \left(\vec{j_s} \times \hat{n}\right) \\ &= 4\ \pi\hat{n} \times \left(\vec{j_s} \times \hat{n}\right) \\ &= 4\ \pi\vec{j_s}. \end{aligned} \tag{3.25}$$

We have supressed the FIDO label for convenience. Recalling the Ohm's law :

$$\vec{J} = \sigma\vec{E} \leftrightarrow \vec{E} = \rho\vec{J}, \tag{3.26}$$

where σ is the conductivity and $\rho = 1/\sigma$ is the resistivity, we see that the equation (3.25) is nothing else than the Ohm's law with a resistivity of $\rho = 4\pi \approx 377\Omega$. We have just shown that using the membrane paradigm, black holes obey Ohm's law.

We will now show that black holes obey the Joule heating law. First, we need to remember the formula of the Poynting flux:

$$\vec{S} = \frac{1}{4\pi}\left(\vec{E} \times \vec{B}\right). \tag{3.27}$$

Using (3.16) and (3.25), we can calculate the formula of the Poyting flux in our case:

$$\begin{aligned}\vec{S} &= \frac{1}{4\pi}\left(\vec{E} \times \vec{B}\right) \\ &= \frac{1}{4\pi}\left(4\pi\vec{j_s} \times \vec{B}\right) \\ &= 4\pi\left(\vec{j_s} \times \vec{j_s} \times \hat{n}\right) \\ &= -j_s^2\rho\hat{n}\end{aligned} \tag{3.28}$$

It can be integrated over the black hole horizon at some fixed time. However, time slicing using fiducial time is not possible because the clocks of different fiducial observers do not necessarily stay synchronized. Therefore, an alternative time frame, such as infinity, must be employed for segmentation purposes. This involves integrating a factor into the integrand, potentially dependent on position, to convert the locally measured energy flux to one at infinity. By strategically selecting the stretched horizon, synchronization among all reference observers can be ensured. Thus, two powers of α, now representing the lapse, are included in the integrand. Subsequently, for a given universal time, t, the power radiated into the black hole, also representing the rate of increase of the black hole's irreducible mass, can be determined as follows:

$$\frac{dM_{irr}}{dt} = -\int \alpha^2\vec{S} \cdot d\vec{A} = \int \alpha^2 j_s^2 \rho dA. \tag{3.29}$$

We have just retrieved the Joule heating law for black holes.

3.2 Gravitation

3.2.1 Equations of motion

For the gravitational action, we take the Einstein-Hilbert action:

$$S_{out}[g^{\mu\nu}] = \frac{1}{16\pi}\int d^4x\ \sqrt{-g}\ R + \frac{1}{8\pi}\oint d^3x\ \sqrt{\pm h}\ K + S_{matter}. \tag{3.30}$$

Here, K is the trace of the extrinsic curvature and R is the Ricci scalar. The inverse metric $g^{\mu\nu}$ has been chosen as the field variable. The surface integral of K is solely performed across the external boundary of spacetime, excluding the stretched horizon. This procedure is necessary to derive the Einstein equations, given that the Ricci scalar encompasses second-order derivatives of $g_{\mu\nu}$.

For now, let's disregard the last two terms and focus on varying the first term. We have:

$$\delta S = \frac{1}{16\pi}\int d^4x \left(\sqrt{-g}\ g^{\mu\nu}\ \delta R_{\mu\nu} + \sqrt{-g}\ R_{\mu\nu}\ \delta g^{\mu\nu} + R\ \delta\sqrt{-g}\right) + \delta S_{matter}. \tag{3.31}$$

We will split the first term it into three terms for easiest computation:

$$(\delta S)_1 = \int d^4x\ \sqrt{-g}\ g^{\mu\nu}\ \delta R_{\mu\nu} \tag{3.32}$$

$$(\delta S)_2 = \int d^4x\ \sqrt{-g}\ R_{\mu\nu}\ \delta g^{\mu\nu} \tag{3.33}$$

$$(\delta S)_3 = \int d^4x\ R\ \delta\sqrt{-g} \tag{3.34}$$

We start with $(\delta S)_1$. To do this, we remark that $\delta R_{\mu\nu}$ is related to $\delta\Gamma^{\rho}_{\nu\mu}$. And because $\delta\Gamma^{\rho}_{\nu\mu}$ is the difference of two connections, it is a tensor [2]. Because it is a tensor, we can take its covariant derivative:

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}. \tag{3.35}$$

With (3.35), we can calculate the variation of $\delta R^{\rho}_{\mu\lambda\nu}$:

$$\delta R^{\rho}_{\mu\lambda\nu} = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma}. \tag{3.36}$$

By adding and substracting the term $\Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu}$ and with (3.35), we can write (3.36):

$$\delta R^{\rho}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \tag{3.37}$$

We now return to $(\delta S)_1$, given by (3.32). It is easy to show that it is given by :

$$\begin{aligned}(\delta S)_1 &= \int d^4x\ \sqrt{-g}\ g^{\mu\nu}\left(\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\right) \\ &= \int d^4x\ \sqrt{-g}\nabla_\sigma\left(g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu})\right). \end{aligned} \tag{3.38}$$

To arrive at the second line, we have renamed λ in σ for the first term in the parentheses, we have renamed $\nu \leftrightarrow \sigma$ in the second term and we have used the metric compatibility.

According to [2]:

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\left(g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right). \tag{3.39}$$

The next step is to put (3.39) in (3.38). And after some calculations that we avoid here for convinience, we arrive at :

$$(\delta S)_1 = \int d^4x\ \sqrt{-g}\ \nabla_\sigma\left(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right). \tag{3.40}$$

We clearly see here that this is a boundary term; hence, it does not affect the equations of motion. We will drop it for now and return to it later.

Now, let's focus on $(\delta S)_2$ and $(\delta S)_3$, given respectively by (3.33) and (3.34). Since we vary the action with respect to $g^{\mu\nu}$, $(\delta S)_2$ is already in the right form. We just need to rework $(\delta S)_3$. To do this, we recall a useful formula:

$$ln(detM) = Tr(lnM). \tag{3.41}$$

Taking the variation of (3.41):

$$\frac{1}{detM}\delta(detM) = Tr(M^{-1}\delta M). \tag{3.42}$$

If we repalce M by $g_{\mu\nu}$, M^{-1} by $g^{\mu\nu}$ and $detM$ by g in (3.42), rearranging the terms, we have:

$$\begin{aligned}\delta g &= g\ (g^{\mu\nu}\delta g_{\mu\nu}) \\ &= -g\ g_{\mu\rho}g_{\nu\sigma}g^{\mu\nu}\delta g^{\rho\sigma} \\ &= -g\ g_{\mu\nu}\delta g^{\mu\nu}\end{aligned} \tag{3.43}$$

where the second line is obtained with the identity $\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$. The last step is to compute :

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu} \end{aligned} \tag{3.44}$$

where the second line is obtained by using (3.43).
It means that, (3.34) is rewritten, with (3.44) by :

$$(\delta S)_3 = -\int d^4x\, \frac{1}{2}\; R\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}. \tag{3.45}$$

We have all the ingredients; let's now derive the equations of motion. Dropping the boundary term $(\delta S)_1$ and using (3.45) and (3.33) in (3.31), we have:

$$\delta S = \frac{1}{16\pi}\int d^4x\, \left(R_{\mu\nu} - \frac{1}{2}\; Rg_{\mu\nu}\right)\sqrt{-g} + \frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \tag{3.46}$$

Dividing all by $\sqrt{-g}$ and recalling the expression of the energy-momentum tensor: $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}$, we can find the EOM by setting $\delta S \approx 0$. Wich leads to :

$$R_{\mu\nu} - \frac{1}{2}\; Rg_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3.47}$$

This is nothing else than Einstein's equation, well-known in general relativity. This is not surprising, as the action is exactly the same as in general relativity.

3.2.2 Surface action

After the derivation of the equations of motion, we should return to the boundary term. This is a very important step of the method because this is the term describing the membrane. This is the term which, we will see later, will lead to the computation of the properties of black holes. To do this, we need to rework the expression of $(\delta S)_1$ given by (3.40). We will start with a small proof that will assist us. We will prove:

$$\nabla_\sigma\left(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right) = \nabla^\sigma\left(\nabla^\lambda(\delta g_{\sigma\lambda}) - g^{\mu\nu}\nabla_\sigma(\delta g_{\mu\nu})\right). \tag{3.48}$$

The unique expression we will use to make this proof is : $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$:

$$\begin{aligned}
\nabla^\sigma \left(\nabla^\lambda(\delta g_{\sigma\lambda}) - g^{\mu\nu}\nabla_\sigma(\delta g_{\mu\nu})\right) &= \nabla^\sigma \left(\nabla^\lambda(-g_{\sigma\alpha}g_{\lambda\beta}\delta g^{\alpha\beta}) - g^{\mu\nu}\nabla_\sigma(-g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta})\right) \\
&= \nabla^\sigma \left(\delta^\nu_\alpha g_{\nu\beta}\nabla_\sigma(\delta g^{\alpha\beta}) - g_{\sigma\alpha}g_{\lambda\beta}\nabla^\lambda(\delta g^{\alpha\beta})\right) \\
&= \nabla^\sigma \left(g_{\alpha\beta}\nabla_\sigma(\delta g^{\alpha\beta}) - g_{\sigma\alpha}\nabla_\beta(\delta g^{\alpha\beta})\right) \\
&= g^{\rho\sigma}\nabla_\rho \left(g_{\alpha\beta}\nabla_\sigma(\delta g^{\alpha\beta}) - g_{\sigma\alpha}\nabla_\beta(\delta g^{\alpha\beta})\right) \\
&= \nabla_\rho \left(g_{\alpha\beta}\nabla^\rho(\delta g^{\alpha\beta}) - \delta^\rho_\alpha\nabla_\beta(\delta g^{\alpha\beta})\right) \\
&= \nabla_\rho \left(g_{\alpha\beta}\nabla^\rho(\delta g^{\alpha\beta}) - \nabla_\beta(\delta g^{\rho\beta})\right) \\
&= \nabla_\sigma \left(g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right) \square.
\end{aligned}$$

Now, let's use (3.48) in (3.40) to rewrite $(\delta S)_1$:

$$\begin{aligned}
(\delta S)_1 &= \int d^4x \ \sqrt{-g} \ \nabla^\sigma \left(\nabla^\lambda(\delta g_{\sigma\lambda} - g^{\mu\nu}\nabla_\sigma(\delta g_{\mu\nu})\right) \\
&= -\int d^3x \ \sqrt{-h} \ n^\sigma \left(\nabla^\lambda(\delta g_{\sigma\lambda}) - h^{\mu\nu}\nabla_\sigma(\delta g_{\mu\nu})\right) \\
&= -\int d^3x \ \sqrt{-h} \ n^\sigma h^{\lambda\mu} \left(\nabla_\mu(\delta g_{\sigma\lambda}) - \nabla_\sigma(\delta g_{\lambda\mu}\right)
\end{aligned} \tag{3.49}$$

where the minus sign appears because n^σ was chosen to be outward pointing to the horizon. And the second line provides from the Gauss' theorem.
We can use the Leibniz rule to write:

$$n^\sigma \nabla_\sigma(\delta g_{\lambda\mu}) = \nabla_\sigma(n^\sigma \delta g_{\lambda\mu}) - \nabla_\mu(n^\sigma)\delta g_{\lambda\mu}. \tag{3.50}$$

We can do the same for $n^\sigma \nabla_\mu(\delta g_{\sigma\lambda})$. Injecting (3.50) in (3.49):

$$\begin{aligned}
(\delta S)_1 &= -\int d^3x \ \sqrt{-h} \ h^{\lambda\mu} \left(\nabla_\mu(n^\sigma \delta g_{\sigma\lambda}) - \nabla_\mu(n^\sigma)\delta g_{\sigma\lambda} - \nabla_\sigma(n^\sigma \delta g_{\lambda\mu}) + \nabla_\sigma(n^\sigma)\delta g_{\lambda\mu}\right) \\
&= \int d^3x \ \sqrt{-h} \ h^{\lambda\mu} \left(\nabla_\sigma(n^\sigma \delta g_{\lambda\mu}) - \delta g_{\lambda\mu}\nabla_\sigma(n^\sigma) - \nabla_\mu(n^\sigma \delta g_{\sigma\lambda}) + \delta g_{\sigma\lambda}\nabla_\mu(n^\sigma)\right).
\end{aligned} \tag{3.51}$$

In [4], it is shown that in the limit where the stretched horizon approaches the null horizon, we have:

$$\int d^3x \ \sqrt{-h} \ h^{\lambda\mu} \left(\nabla_\sigma(n^\sigma \delta g_{\lambda\mu}) - \nabla_\mu(n^\sigma \delta g_{\sigma\lambda})\right) = 0. \tag{3.52}$$

We just need to put (3.52) in (3.51):

$$(\delta S)_1 = \int d^3x \sqrt{-h}\, h^{\lambda\mu} \left(\delta g_{\sigma\lambda} \nabla_\mu (n^\sigma) - \delta g_{\lambda\mu} \nabla_\sigma (n^\sigma)\right)$$
$$= \int d^3x \sqrt{-h}\, h^{\lambda\mu} \left(g_{\lambda\rho} g_{\mu\alpha} \delta g^{\rho\alpha} \nabla_\sigma (n^\sigma) - g_{\sigma\rho} g_{\lambda\alpha} \delta g^{\rho\alpha} \nabla_\mu (n^\sigma)\right) \tag{3.53}$$

where the second line comes from the identity $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$. With (2.5), we can write some expressions:

$$K^{\mu\nu} = h^{\nu\rho} \nabla_\rho n^\mu \tag{3.54}$$
$$K_{\mu\nu} = h_{\nu\rho} \nabla^\rho n_\mu = h^\rho_\nu \nabla_\rho n_\mu \tag{3.55}$$
$$K = g_{\mu\nu} K^{\mu\nu} = g_{\mu\nu} h^{\nu\rho} \nabla_\rho n^\mu = h^\rho_\mu \nabla_\rho n^\mu = \nabla_\mu n^\mu. \tag{3.56}$$

With expressions (3.54) to (3.56), we arrive at the final expression of the surface action :

$$(\delta S)_1 = \delta S_{out}[g^{\mu\nu}] = \frac{1}{16\pi} \int d^3x \sqrt{-h} \left(h_{\rho\alpha} K \delta g^{\rho\alpha} - h^{\lambda\mu} g_{\sigma\rho} g_{\lambda\alpha} \delta g^{\rho\alpha} \nabla_\mu (n^\sigma)\right)$$
$$= \frac{1}{16\pi} \int d^3x \sqrt{-h} \left(h_{\rho\alpha} K \delta g^{\rho\alpha} - h^\mu_\alpha g_{\sigma\rho} \nabla_\mu (n^\sigma) \delta g^{\rho\alpha}\right)$$
$$= \int d^3x \sqrt{-h} \left(K h_{\rho\alpha} - K_{\rho\alpha}\right) \delta g^{\rho\alpha}. \tag{3.57}$$

We have arived at an expressions with only stretched horizon tensors. That means that, from (2.4): $\delta g^{\mu\nu} = \delta h^{\mu\nu} + \delta n^\mu n^\nu + n^\mu \delta n^\nu = \delta h^{\mu\nu}$, the normal vector contribute nothing (it is normal to the stretched horizon). Exactly like we have done for the section 3.1 (electromagnetic case), we add 'by hand' a surface source term to cancel the residual surface term. The expression of this term, already variated, should be :

$$\delta S_{surf}[h^{\mu\nu}] = -\frac{1}{2} \int d^3x \sqrt{-h}\; t_{s\ \mu\nu} \delta h^{\mu\nu} \tag{3.58}$$

where $t_{s\ \mu\nu}$ is the membrane stress tensor. A brief comparison of (3.58) with (3.57) provides us the expression of the membrane stress tensor in our case:

$$t_s^{\mu\nu} = \frac{1}{8\pi} \left(K h^{\mu\nu} - K^{\mu\nu}\right). \tag{3.59}$$

We can see that this is exactly the Israel junction condition, slightly rewritten. In [3], we clearly see that the Israel junction condition can be expressed as:

$$t_s^{\mu\nu} = \frac{1}{8\pi} \left([K] h^{\mu\nu} - [K]^{\mu\nu}\right) \tag{3.60}$$

where $[K] = K_+ - K_-$. $[K]$ represents the difference in the extrinsic curvature of the stretched horizon between : its integration into the broader universe and its integration into the spacetime within the black hole.
With this explanation, one can see that :

$$K_-^{\mu\nu} = 0, \tag{3.61}$$

As just stated, equations (3.59) and (3.60) are equal. Physically, this implies that the extrinsic curvature of the stretched horizon within the spacetime of the black hole is null. Consequently, the interior of the stretched horizon is flat space, and can therefore be described by Minkowskian space.

In [3], the Gauss-Codazzi equations are introduced:

$$K^{\mu\nu}{}_{|\nu} - K_{|\mu} = h^\mu_\rho R^{\rho\sigma} n_\sigma \tag{3.62}$$

where $R^{\rho\sigma}$ is the Ricci tensor. We can rewrite these equations :

$$K^\mu_{\nu\ |\mu} - K_{|\nu} = R_{\rho\sigma} n^\sigma h^\rho_\nu. \tag{3.63}$$

We can use this to compute the 3-covariant derivative of equation (3.59). Let's do it :

$$\begin{aligned}
t^{\mu\nu}_{s\ |\nu} &= \frac{1}{8\pi}\left(Kh^{\mu\nu} - K^{\mu\nu}\right)_{|\nu} \\
&= \frac{1}{8\pi}\Big(\left(Kh^{\mu\nu}\right)_{|\nu} - \left(K^{\mu\nu}\right)_{|\nu}\Big) \\
&= \frac{1}{8\pi}\Big(K_{|\nu}h^{\mu\nu} - K^{\mu\nu}{}_{|\nu}\Big) \\
&= \frac{1}{8\pi}\Big(K^{|\mu} - K^{\mu\nu}{}_{|\nu}\Big) \\
&= -\frac{1}{8\pi}\left(h^\mu_\rho R^{\rho\sigma} n_\sigma\right) \\
&= -h^\mu_\rho T^{\rho\sigma} n_\sigma
\end{aligned} \tag{3.64}$$

where $T^{\rho\sigma} = \frac{R^{\rho\sigma}}{8\pi}$ is the stress-energy tensor. The first lines are just coming from the linearity of the covariant derivative and the fifth line is obtained by using (3.62). What we have just writed is simply a rewriting of the EOM (3.47).

We now arrive at the most significant part of this report: computing the thermodynamic properties of black holes. We first demonstrate that equations (3.59) and

(3.64) imply that the stretched horizon obeys the Navier-Stokes equation, thus can be considered as a fluid membrane.
As mentioned in section 2, we can equate αU^μ and αn^μ in the limit as we set α to zero. In this limit, both αU^μ and αn^μ approach the null generator l^μ at the point on the true horizon. Therefore, we can express the components of the extrinsic curvature K^μ_ν in terms of the extrinsic curvature of a spacelike 2-section of the stretched horizon k^A_B and the surface gravity κ:

$$\begin{aligned} U^\rho \nabla_\sigma n^\mu \to \alpha^{-1} l^\sigma \nabla_\sigma \alpha^{-1} l^\mu &= \alpha^{-2} l^\sigma \nabla_\sigma l^\mu \\ &\equiv \alpha^{-2} \kappa_H l^\mu \end{aligned} \tag{3.65}$$

where $\kappa_H \equiv l^\sigma \nabla_\sigma \equiv \alpha\kappa$ is the renormalized surface gravity at the horizon and κ is just the surface gravity at the horizon.
Equation (3.65) implies :

$$\begin{aligned} K^\nu_\mu U^\mu U_\nu &= h^\rho_\mu \nabla_\rho n^\nu U^\mu U_\nu \\ &\equiv -\kappa = -\alpha^{-1}\kappa_H. \end{aligned} \tag{3.66}$$

Also, we can see that :

$$\begin{aligned} K^A_U &= \gamma^A_\mu K^\mu_\nu U^\nu \\ &= \gamma^A_\mu h^\rho_\nu \nabla_\rho n^\mu U^\nu = 0. \end{aligned} \tag{3.67}$$

With the same arguments, we can write some other expression:

$$\begin{aligned} \gamma^\rho_A \nabla_\rho n^\nu \to \alpha^{-1} \gamma^\rho_A \nabla_\rho l^\nu \implies K^B_A &= \gamma^\mu_A K^\nu_\mu \gamma^B_\nu \\ &= \gamma^\mu_A \gamma^\rho_\mu \nabla_\rho n^\nu \gamma^B_\nu \\ &\to \alpha^{-1} \gamma^\mu_A h^\rho_\mu \nabla_\rho l^B \\ &= \alpha^{-1} \gamma^\mu_A l^B{}_{||\mu} \\ &= \alpha^{-1} k^B_A. \end{aligned} \tag{3.68}$$

Here, k^B_A represents the extrinsic curvature of a spacelike 2-section of the true horizon, not the stretched horizon. This distinction arises because we've set the limit of α to zero, signifying that the stretched horizon coincides with the true horizon, as mentioned in the introduction.
For more convenience, we introduce a little results [2]:

$$\mathcal{L}_V \; g_{\mu\nu} = 2 \; \nabla_{(\mu} V_{\nu)} \tag{3.69}$$

where $\mathcal{L}_V$ is the Lie derivative in the V direction.
In our case, we can write, with (3.69):

$$\begin{aligned}\mathcal{L}_{l^\mu}\ \gamma_{AB} &= l^\mu_{A\ ||B} + l^\mu_{B\ ||A} \\ &= 2l^\mu_{B\ ||A}.\end{aligned} \tag{3.70}$$

When we have derived (3.68), we see that in the last line we have juste retrieved $k_A^B = \gamma^\mu_A l^B_{||\mu}$. Thanks to this definition and to (3.70) we have :

$$k_{AB} = \gamma^\sigma_A l_{B\ ||\sigma} = \frac{1}{2}\mathcal{L}_{l^\mu}\gamma_{AB}. \tag{3.71}$$

What we have done is just rewriting the extrinsic curvature k_{AB} with a more compact way, using the commonly known Lie derivative.
We are nearing our goal. All these computations will help us find the expression of the stress-energy tensor of the spacelike 2-section of the stretched horizon, and ultimately, in the limit, of the true horizon. It is introduced in [7] that we can decompose the extrinsic curvature k_{AB} into a traceless part, σ_{AB} (the shear), and a trace, θ (the expansion of the world lines of nearby horizon surface elements [4]), leading to:

$$k_{AB} = \sigma_{AB} + \frac{1}{2}\gamma_{AB}\theta. \tag{3.72}$$

Owing to (3.59), we can write the stress energy tensor for a spacelike 2-section of the stretched horizon:

$$t_s^{AB} = \frac{1}{8\pi}\left(K\gamma^{AB} - k^{AB}\right). \tag{3.73}$$

We need to calculate what is the expression of K :

$$\begin{aligned}K &= K^{\mu\nu}g_{\mu\nu} \\ &= K^{\mu\nu}\gamma_{\mu\nu} + K^{\mu\nu}n_\mu n_\nu - K^{\mu\nu}U_\mu U_\nu \\ &= \theta + \kappa.\end{aligned} \tag{3.74}$$

To obtain the second line, we have used (2.4). The last line is obtained owing to (3.72), (3.66) and (2.5).
Injecting (3.74) and (3.72) in (3.73), we arrive at the final expression of the stress energy tensor of a spacelike 2-section of the stretched horizon:

$$\begin{aligned}t_s^{AB} &= \frac{1}{8\pi}\left((\theta+\kappa)\gamma^{AB} - \sigma^{AB} - \frac{1}{2}\gamma^{AB}\theta\right) \\ &= \frac{1}{8\pi}\left(-\sigma^{AB} + \gamma^{AB}\left(\frac{1}{2}\theta + \kappa\right)\right).\end{aligned} \tag{3.75}$$

Always in [7], it is shown that t_s^{AB} can be written as the equation for the stress of a two-dimensional viscous Newtonian fluid:

$$t_s^{AB} = 2\eta\sigma^{AB} + (-p + \zeta\theta)\,\gamma^{AB}, \tag{3.76}$$

permitting us to identify a pressure $p = \frac{\kappa}{8\pi}$, a shear viscosity $\eta = \frac{1}{16\pi}$ and a bulk viscosity $\zeta = -\frac{1}{16\pi}$. Thus, we can liken the horizon to a two-dimensional dynamic fluid or membrane. It's noteworthy that, unlike conventional fluids, this membrane possesses negative bulk viscosity. Typically, this would imply vulnerability to perturbations triggering expansion or contraction. However, in the context of a null hypersurface, this trait can be seen as a natural inclination towards expansion or contraction [6]. In the following sections, we will illustrate how this particular instability is supplanted by a different form of instability for the horizon [4].
Like [4], we can introduce the A-momentum density:

$$t_s{}^{\nu}{}_{\mu}\gamma_A^{\mu}U_\nu = t_s^U{}_A \equiv \pi_A. \tag{3.77}$$

Using (3.64) with (3.77) :

$$\begin{aligned}
\nabla_0\pi_A &= \nabla_0\left(t_\mu^\nu\gamma_A^\mu U_\nu\right)\\
&= \partial_0\left(t_\mu^0\gamma_A^\mu U_0\right)\\
&= \partial_0\left(t_\mu^0\gamma_A^\mu\right)\\
&= -t^B_{\mu||B}\gamma_A^\mu + t^\nu_{\mu|\nu}\gamma_A^\mu\\
&= -t^B_{\mu||B}\gamma_A^\mu - h_{\rho\mu}T^{\sigma\rho}n_\sigma\gamma_A^\mu\\
&= -t^B_{\mu||B}\gamma_A^\mu - T^\sigma_\mu n_\sigma\gamma_A^\mu\\
&= -t^B{}_{A||B} - \gamma_A^\mu T^\sigma_\mu n_\sigma,
\end{aligned} \tag{3.78}$$

where the second line comes because this is a scalar in the parentheses, the third comes from $U^0 = 1$, the fourth line comes from the Leibniz rule, the fifth is derived from the expression (3.64), and the last lines are just a rewrite of the previous line. A few more steps of calculation lead us to:

$$\frac{\partial\pi_A}{\partial\tau} = \mathcal{L}_\tau\pi_A = -\nabla_A p + \zeta\nabla_A\theta + 2\eta\sigma_A^B{}_{||B} - T_A^n \tag{3.79}$$

where $-T_A^n = -\gamma_A^\mu T_\mu^\sigma n_\sigma$ is the flux of A-momentum into the black hole and $\mathcal{L}_\tau\pi_A = \mathcal{L}_0\pi_A$ is the Lie derivative with respect to proper time, because we have chosen the proper time to parametrize world lines. What we have written is nothing else than the Navier-Stokes equation for the stretched horizon. This allows us to once again identify the stretched horizon (and in the limit, the true horizon) as a membrane or a dynamical fluid in two dimensions. This is a remarkable result. With this, we can now compute the expression of the temperature and entropy, broadly speaking, the thermodynamic properties of the horizon of a black hole without knowing anything inside the black hole.

Like for the A-momentum density, we can introduce the U-momentum density :

$$\begin{aligned}\Sigma &= t_{s\ \nu}^{\ \mu} U_\mu U^\nu \\ &= \frac{1}{8\pi}\left((\theta+\kappa)\,\gamma_\nu^\mu U_\mu U^\nu - (\theta+\kappa) - K_\nu^\mu U_\mu U^\nu\right) \\ &= \left((\theta+\kappa)\,\gamma_\nu^\mu U_\mu U^\nu - \theta\right) \\ &= \frac{-\theta}{8\pi} \end{aligned} \tag{3.80}$$

where the second line comes from (3.59), the third from $K_\nu^\mu U_\mu U^\nu = \kappa$ and the last from $\gamma_\nu^\mu U_\mu U^\nu = 0$ (easily verified with (2.4)). After some step of computation, similar to the step used to obtain (3.79), inserting (3.80) in (3.79) gives us :

$$\mathcal{L}_\tau \Sigma + \theta\Sigma = -p\theta + \zeta\theta^2 + 2\eta\sigma_{AB}\sigma^{AB} + T_\nu^\mu n_\mu U_\nu. \tag{3.81}$$

This is the focusing equation for a null geodesic congruence. Given our previous analogies with dynamical fluids, this equation can be interpreted as the equation of energy conservation. In thermodynamics, it corresponds to the heat transfer equation for a dynamical fluid in two dimensions.

Last but not least, we will find the expression of the Temperature and the Entropy. To do this, we express the expansion of the fluids in terms of ΔA, the area of a patch:

$$\theta = \frac{1}{\Delta A}\frac{d\Delta A}{d\tau}. \tag{3.82}$$

If we postulate that the temperature can be written like:

$$T = \frac{\hbar}{8\pi k_B \eta}\kappa, \tag{3.83}$$

where k_B is the Boltzmann constant, $\hbar$ is the Planck constant and η is a proportionality constant. In the same way, if we postulate that the entropy can be written:

$$S = \eta\frac{k_B}{\hbar}A, \tag{3.84}$$

we see that putting (3.83) and (3.84) with (3.82) and (3.80) in (3.81), after some steps of computation:

$$T\left(\frac{d\Delta S}{d\tau}-\frac{1}{\kappa}\frac{d^2\Delta S}{d\tau^2}\right)=\left(\zeta\theta^2+2\eta\sigma_{AB}\sigma^{AB}+T^{\mu}_{\nu}n_{\mu}U^{\nu}\right)\Delta A. \tag{3.85}$$

This is the exact form of the heat transfer equation. And the analogy between the horizon and the fluid permit us to find the temperature and the entropy of black holes without knowing anything inside it.

4 Demonstration

We will apply our results to a Reissner-Nordström (RN) black hole. The metric of a RN black hole is :

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{4.1}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{4.2}$$

We want to know what is n_μ and α. To do this we use the definition [poisson]:

$$n_\mu = \frac{\varepsilon\, \partial_\mu \Phi}{|g^{\alpha\beta}\partial_\alpha \Phi \partial_\beta \Phi|^{1/2}}, \tag{4.3}$$

where $\varepsilon = n^\mu n_\mu = +1$ for timelike hypersurfaces (our case with the stretched horizon) and Φ is the restriction on the coordinates to obtain our hypersurface.
For us, the stretched horizon has a constant r, meaning that we can set:

$$\Phi = r. \tag{4.4}$$

So, we can find the expression of n_μ by putting (4.4) in (4.3):

$$\begin{aligned} n_\mu &= \frac{\partial_\mu r}{|g^{\alpha\beta}\partial_\alpha r \partial_\beta r|^{1/2}} \\ &= \frac{\partial_\mu r}{|g^{rr}|^{1/2}}. \end{aligned} \tag{4.5}$$

But, if we look at our metric (4.1), we see that $g^{rr} = f(r)$. That means that we can write (4.5) :

$$n_\mu = f^{-1/2}(r)\partial_\mu r = f^{-1/2}(r)(dr)_\mu. \tag{4.6}$$

We have computed the expression of n_μ. It is easy to verify that $n^\mu n_\mu = 1$ as expected. The results (4.6) permits us to know what is the expression of α:

$$\alpha = f^{1/2} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2}. \tag{4.7}$$

And so $n_\mu = \alpha^{-1}(dr)_\mu$. We can do same things to retrieve the expression of U_μ, we have :

$$U_\mu = -\alpha(dt)_\mu. \tag{4.8}$$

We can compute the temperature of a RN black hole with the definition (3.83). In this equation, there is the surface gravity κ.
The surface gravity is just the gravitational acceleration of some test object at the surface of the body studied (planet, star, blackhole,...). Here we will use a more convenient and more common equation to compute it:

$$\kappa^2 = -\frac{1}{2}\nabla^\mu \xi^\nu \nabla_\mu \xi_\nu, \tag{4.9}$$

where ξ^ν is the killing vector. In our case the killing vector is $\xi^\nu = \partial_t = (1,0,0,0)$. We work in the Levi-Civita connection, meaning that the covariant derivative of an arbitrary vector field A^ν is:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho, \tag{4.10}$$

with the Christoffel symbol :

$$\Gamma^\nu_{\mu\rho} = \frac{1}{2} g^{\nu\alpha} \left(\partial_\mu g_{\rho\alpha} + \partial_\rho g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right). \tag{4.11}$$

Let's compute the surface gravity with the expression (4.9):

$$\begin{aligned}
\kappa^2 &= -\frac{1}{2}\nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \\
&= -\frac{1}{2}(g^{\mu\rho}\nabla_\rho \xi^\nu)(g_{\nu\sigma}\nabla_\mu \xi^\sigma). \\
&= -\frac{1}{2}(g^{\mu\rho}\nabla_\rho \xi^t)(g_{tt}\nabla_\mu \xi^t).
\end{aligned} \tag{4.12}$$

where the last line arises because ξ^t is the only non zero component of the killing vector. We will skip the details here but it is easy to show that:

$$\nabla_\mu \xi^t = \Gamma^t_{tr} = \frac{1}{2}\frac{\partial_\mu f}{f}, \tag{4.13}$$

because $\xi^t = 1$. And because f only depends on r, we have, by using the metric (4.1) and injecting (4.13) in (4.12):

$$\kappa^2 = \frac{1}{4}\left(\frac{df}{dr}\right)^2. \tag{4.14}$$

We have skip some step of computation but these are only algebra. Obviously,

$$\kappa = \frac{1}{2}\frac{df}{dr}. \tag{4.15}$$

Before calculating the derivative of f, we will express f in terms of r_+ and r_- wich are respectively the radius of the outer and the the inner horizons of the black hole. They are obtained by solving the equation $f(r) = 0$ because these are the points where the metric diverges. We directly show that :

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{4.16}$$

And if we are approching near r_+ (which is reasonnable because we approache the black hole by its exterior), we can rewrite f in terms of r_+ and r_- :

$$f(r) = \frac{(r_+ - r_-)(r - r_+)}{r_+^2}. \tag{4.17}$$

By putting (4.17) in (4.15), we arrive at the temperature of a RN black hole:

$$T = \frac{\hbar}{16\pi k_B \eta}\frac{r_+ - r_-}{r_+^2}. \tag{4.18}$$

This is a very simple equation. And the best part of the story, when we put ourselves in the case of an extremal RN black hole, where $r_+ = r_-$, we directly see that:

$$T = 0. \tag{4.19}$$

This is the common result for an extremal black hole, using other methods. That means that our methods is fully consistent and reproduces known results.
It is not necessary to compute the entropy S for this particular case because we see, with (3.84), that the entropy only depends on the area A.

5 Conclusion

During this internship, we have demonstrated that we can compute the most important properties of black holes without any knowledge of the interior, solely relying on classical mechanics and the action principle. While we have performed all computations for a general black hole, the same approach can be applied to specific black holes, even those not in four dimensions. All we require is an action for the black hole.
We have seen an example of a non general black hole with the section 4 for the Reissner-Nordström black hole. In future work we can see if the Membrane Paradigm method can reproduces known results for near-extremal black hole.
A near-extremal black hole is one that is very close to the extremal limit, where it possesses the maximum possible charge or angular momentum allowed by the laws of physics. These black holes are of interest in theoretical physics because they exhibit unique properties and behaviors. This type of black holes has a temperature very close to zero but not totally zero.

For other future direction we can also apply this method to a BTZ black hole wich is a three-dimensional solution to Einstein's equations of general relativity with negative cosmological constant. It is named after its discoverers, Banados, Teitelboim, and Zanelli. The BTZ black hole has the interesting property of being an exact solution to the equations of motion, providing valuable insights into the behavior of black holes in three dimensions, which can be useful in understanding certain aspects of black hole physics and quantum gravity.

What we have tried to show is that we can understand and compute some of the best properties of a black hole by a simple model, just knowing the action and nothing inside the black hole. This model can help us to understand how black holes works and can help us to understand a little bit more on physics and on the world in wich we live.

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